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Integral Mathematics
AN AQAL APPROACH

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This article lays the foundations for a comprehensive formulation of Integral Mathematics. I introduce a number of definitions for an Integral Mathematics, including the perspective of Integral Mathematics in the context of a distinct discipline, Ken Wilber's symbolic Integral Mathematics of primordial perspectives, and Integral Mathematics as a developmental line. I then focus on the perspective of Integral Mathematics as a distinct discipline and show that the "division" between the worlds of pure and applied mathematics is parallel to the distinction between the Left- and Right-Hand quadrants. I give a four-quadrant analysis for Integral Mathematics as a distinct discipline by applying the quadrants to Perfect Numbers from Recreational Number Theory. Finally, I give an example from Group Theory that illustrates how Integral Mathematics may be applied to explore shifts in levels of consciousness through meditation.

Integral Mathematics Perspectives

Mathematics is both an art form and a scientific discipline. When philosopher Ken Wilber writes about the differentiation of "The Big Three" (i.e., Art, Morals, and Science), mathematics is in the unique position of being both a subjective art form as well as an objective science.[1] Mathematicians might recognize this as the division of the field into pure mathematics and applied mathematics. Author Jerry King, in *The Art of Mathematics*, stresses that these two disciplines of mathematics are as far apart as the mystical poet and the objective scientist.[2] However, King also calls for the integration of pure mathematical thinking and pragmatic mathematical application. In other words, King is asking that the realms of art and science be integrated in a unified mathematics, quite analogous to the Integral model's quest for integration in other disciplines of knowledge, including psychology, spirituality, medicine, law, politics, government, education, business, and so on. In this article, I would like to propose that we add mathematics to the growing list of emerging Integral disciplines.

We can view Integral Mathematics from a variety of perspectives, including the following:

1. Ken Wilber's "Calculus of Indigenous Perspectives"[3]
2. Mathematics as an independent line of development[4]
3. A four-quadrant analysis applied to mathematics as a discipline, focusing upon the various branches of mathematical study.
4. A four-quadrant analysis applied to a particular age group.
5. A four-quadrant analysis applied to mathematical research.
6. Integral Mathematics as a bona-fide mathematical discipline in itself.

Each of the above perspectives of Integral Mathematics is a legitimate approach, but my main concentration in this article will be on perspective #3 (a four-quadrant analysis applied to mathematics as a discipline, focusing upon the various branches of mathematical study). I will describe how the pure mathematics disciplines of number theory and group theory (a group theory/consciousness problem is described in the Appendix) can be extended into an applied mathematics context that involves all four quadrants in Integral Theory (the Upper Left, the Upper Right, the Lower Left, and the Lower Right). The disciplines of pure and applied mathematics are related to the division of the Upper-Left (UL) and Upper-Right (UR) quadrants. (Here we are utilizing a four-quadrant perspective on the branches of mathematics, which is technically defined as a "quadrivium.")

In the Upper-Left quadrant, we have such things as the cognitive line of development as well as a continuum of mathematical knowledge that develops through levels in a "holarchical" fashion.[5] For example, when fourth graders learn their multiplication and division facts, they do not forget these facts (hopefully) when they learn high school algebra. Rather, their ingrained arithmetic skills become the foundation for solving higher-level mathematical problems.

Of course, our amazing technological advances in recent years make it unnecessary to retain many formerly necessary mathematical skills (arithmetic, algebra, and so on); although I contend that it is intrinsically beneficial for us to retain the mathematical knowledge upon which our technology is based. However, incorporating these technological developments into Integral Mathematics brings us to the Lower-Right (LR) quadrant of the Integral model. For example, as a pure mathematician researcher in the field of Algebraic Number Theory, I have learned to appreciate the tremendous usefulness of the mathematical computer software program "Pari" in furnishing me with extremely complicated examples of theoretical mathematical results I have proved. I would have never been able to come up with these examples without the use of this technology, and I view my pure mathematics research in this context as an example of Integral Mathematics perspective #5. (For the interested reader, I have a paper that will appear in the *Ramanujan Journal* that weaves my pure mathematics results with my technologically-based examples.)[6] For me, it has been truly enlightening to blend technology with pure mathematics, and when I presented my results to the Maine/Quebec Number Theory Conference in 2002, the response was that it was very refreshing to see this kind of balance between theory and technology.[7]

The Lower-Left (LL) quadrant for Integral Mathematics can be described as the cultural "We" space, where an Integral view of mathematics can be shared with others. A prime (excuse the mathematical pun) example of my own experience with Integral Mathematics involves the work I have done in both elementary schools and college courses, where I have taught from my book *Numberama: Recreational Number Theory in the School System*.[8] In the context of Recreational Number Theory, which involves exploring the intriguing patterns of our number system as a recreational pastime, I believe that this exploration is a unique learning experience for children and liberal arts students in the realm of pure mathematics (UL). At the same time, I make sure that my elementary students diligently practice their multiplication and division skills and that all my students learn how to make use of their age appropriate technology tools (from arithmetic to scientific calculators), in order to do the necessary trial and error work of discovering these patterns. (These activities focus on the UR and LR.)

It is important to keep in mind that here I am addressing Integral Mathematics perspective #3 (i.e., mathematics as a particular subject and discipline) when I view the calculation skills as an UR activity. In Integral Mathematics perspective #2, where the focus is upon the entire range of mathematics as a developmental line within people, all mathematical thinking (including calculation skills) would be considered an UL activity, while UR activities would consist of such things as the observable behaviors and brain wave states of students. The LR aspects would include mathematical symbols and written language, the classroom settings where mathematical communication takes place, the actual external communications of language utilized to discuss mathematics, etc.

For older children and college students, their exploration and discovery of mathematical patterns eventually results in concrete algebraic formulas.[9] There is much collaboration amongst students in working together to explore my problems, and my book includes twenty games that teachers, children, and parents can play together to further practice their skills.[10] For me, it is a way to bridge the gap between my own ivory tower mathematical interests and the pragmatic view of mathematics that most people have. Another example of the Lower-Left quadrant in Integral Mathematics would be something like the "Family Math" workshops I have offered for parents and children working together on these Recreational Number Theory problems. In this Family Math context, we have a strong LL activity where families work with each other in collaboration and mutual understanding to explore Recreational Number Theory problems (UL), with much arithmetical calculation practice (UR), in conjunction with classroom settings, verbal exchanges of the mathematics involved, and technology in the form of calculators and occasionally the internet (LR).

We thus have all four quadrants well represented in Integral Mathematics: the Upper Left for the intrinsic, artistic, pure mathematical experience of exploring and discovering patterns of numbers; the Upper Right for the objective, disciplined, arithmetic skills practice with eventual concrete algebraic formulas; the Lower Left for the collaboration of students, parents, and teachers working together to discover these intrinsic number theory patterns; and the Lower Right for the outward forms of communication, physical resources, and the use of technology in the form of calculators and computers.

To illustrate how I utilize an Integral approach to Recreational Number Theory when teaching the joys of mathematics, I will focus upon the example of Perfect Numbers.[11] Although I have taught Perfect Numbers in an Integral Mathematics context to all age groups, my description in a later section is particularly well suited for the age group of upper elementary school children, which is an illustration of Integral Mathematics perspective #4.

Let me emphasize that my discussion of Integral Mathematics is not limited to Wilber's calculus of indigenous perspectives (Integral Mathematics perspective #1). Wilber's approach is essentially a mathematical symbolic language used to describe first-person, second-person, and third-person perspectives, with further perspectives on the horizon.[12]

My main goal, however, is to demonstrate how the four quadrants of the Integral model can be applied to mathematics as a discipline of study, focusing on pure mathematics in the context of UL intrinsic mathematical thinking. I would also like to add that Integral Mathematics perspective #6, which views Integral Mathematics as its own mathematical discipline, actually includes both my Perfect Numbers example in a later section of this paper as well as my Group Theory/Consciousness example in the Appendix. To give two well known examples from applied mathematics, we will briefly look at the Pythagorean Theorem from high school Trigonometry and the Fundamental Theorem of Calculus from first year math and science majors.

Mathematics as a Discipline: UL & UR

I believe that virtually all branches of mathematics have components in the lower or collective quadrivia (plural of quadrivium): intersubjective LL and interobjective LR. Through cultural collaboration amongst mathematicians and scientists and the tremendously widespread use of the internet, computer programs, textbooks, research papers, seminars, and classroom settings, it seems quite evident that the lower quadrivia are well represented in virtually every field of mathematics. However, when we study the upper or individual quadrivia of mathematics in the context of the pure and applied mathematics divisions (i.e., the UL and UR, respectively), the picture is not quite as simple. The particular classification scheme that I have devised can also be described in Ken Wilber's symbolic language of indigenous perspectives (Integral Mathematics perspective #1), where pure mathematics would focus on a first-person perspective and applied mathematics would focus on a third-person perspective.

PURE MATHEMATICS (UL)

Number Theory

Abstract Algebra

Topology

APPLIED MATHEMATICS (UR)

Statistics

Differential Equations

Pre-Calculus (Analytic Geometry, Trigonometry, High school Algebra, Arithmetic)

PURE AND APPLIED MATHEMATICS (UL & UR)

Calculus

Analysis (Real & Complex)

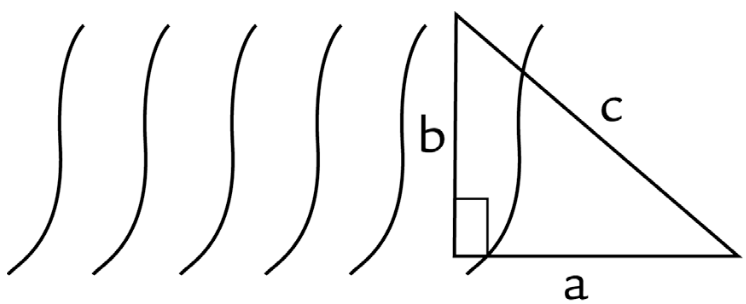
Probability

Geometry (Euclidean & Projective)

Set Theory

Figure 1. Disciplines of Mathematics: UL & UR

To illustrate Integral Mathematics perspective #6 (Integral Mathematics itself as a mathematics discipline), I will take a look at two well-known applied mathematics problems from Trigonometry and Calculus. To begin with, we can measure the width of a river without crossing it by applying the Pythagorean Theorem from Trigonometry, which says that in a right triangle (a triangle with a perpendicular corner), the square of the side opposite the right angle (the hypotenuse) is equal to the sum of the squares of the two other sides. This is generally described algebraically as c^2 = a^2 + b^2, where the symbol ^ denotes an exponent and thus 5^2 = 5 X 5 = 25 (see figure 2). The crossing the river problem starts out as a highly pragmatic example from Trigonometry in the UR quadrivium, but one can study how to prove the Pythagorean Theorem using logical-mathematical thinking, which involves the UL quadrivium. Whether in a classroom setting or out surveying the shores of the river, one can culturally connect with others thru interactive learning communities, which involves the LL quadrivium. Finally, one can employ various technologies such as surveying equipment and classroom calculators, as well as engage in verbal-mathematical exchanges, all of which highlights the LR quadrivium.



$$a^2 + b^2 = c^2$$

If a = 50 ft. and c = 130 ft.,
then the river with side b
can be calculated as $50^2 + b^2 = 130^2$ or
$b^2 = 130^2 - 50^2 = 16{,}900 - 2{,}500 = 14{,}400$
and therefore $b = \sqrt{14{,}400} = 120$ ft.

Figure 2. “Crossing the River” Problem

A similar argument can be made using Calculus for the problem of finding the area under the parabolic bell-shaped curve y = x^2, for the range from x = 2 to x = 7 (see figure 3). A parabola can always be described by a quadratic (highest exponent of 2) equation in algebra, which is a common representation for many kinds of scientific problems, ranging from measuring the height of an object thrown from the ground to the probability distribution of so-called "normal" distributions from Statistics. Once again, we start with an UR problem. As it turns out, we can estimate this area by measuring the area of a number of small rectangles inside the parabola. As the sum of the areas of our rectangles approach the whole space of the parabola by increasing in number and reducing in size, the better our area approximation will be. However, we can get the exact area of the parabola by using what is known as The Fundamental Theorem of Calculus, which relates the theory of anti-derivatives to finding the area of various geometric curves. For our purposes right now, what is important is that the Fundamental Theorem of Calculus can be proven, and this is generally done for first year math and science major students. The LL and LR quadrivia once again can be utilized in various teaching/learning scenarios with a rich array of educational and technical resources.

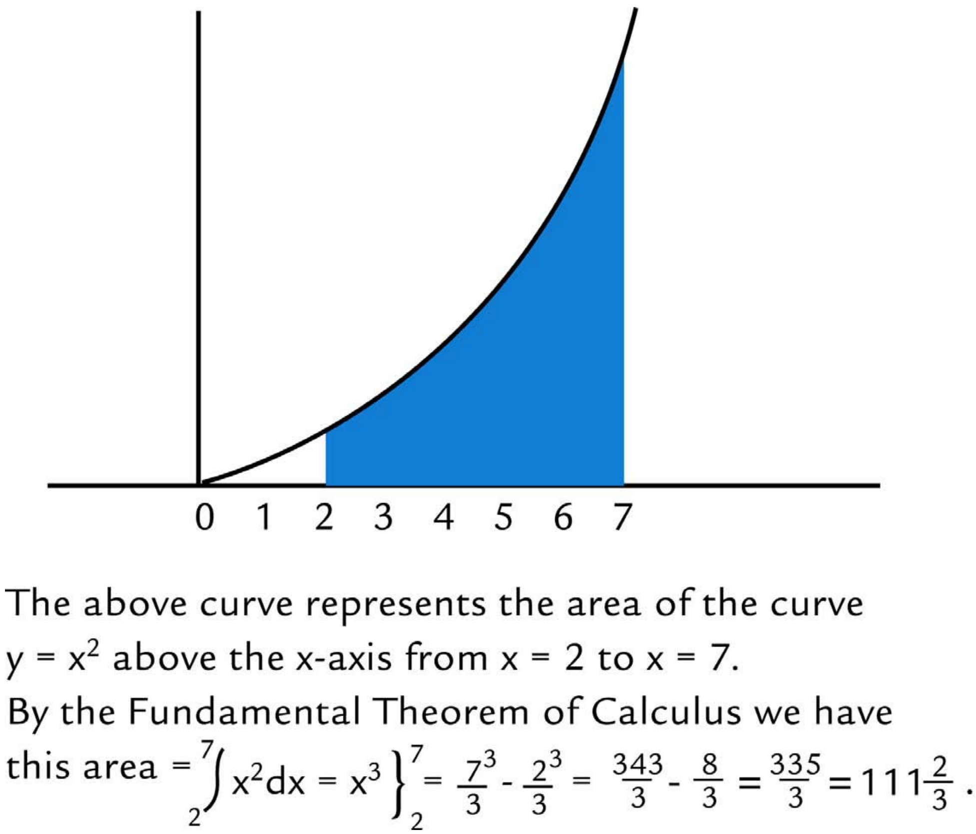


Figure 3. "Area of Parabola" Problem

From this brief glimpse into the world of well-known, applied mathematics, together with the examples I will describe from pure mathematics in this paper's section on Perfect Numbers and the Appendix, it appears that there is much potential to view Integral Mathematics as a separate field of mathematical study.

The Developmental Line of Mathematics

One way of approaching the developmental line of mathematical thinking is to use Piaget's levels of cognitive development: sensorimotor, preoperational, concrete operations, and formal operations.[13] According to Piaget, young children before the age of 6 or 7 are in the preoperational stage and have not yet developed "number conservation," meaning that their sense of number is not intact. For example, if one increases the distance between a given number of objects, effectively spreading them out, this usually causes children to believe that there are more objects present when they are spread out than when they are bunched close together, despite the fact that the number of objects remains the same.[14] However, it should be noted that there is also disagreement with Piaget's conclusion that young children do not have a real number sense given that recent research in neuropsychology and brain physiology suggests that young children may not understand the instructions of the experimenter and may have a different interpretation of what is meant by "more," "less," etc.[15] (Although Piaget's actual levels of cognitive development are universally accepted.) At any rate, let us assume for the moment that children between the ages of 7 and 11 generally enter Piaget's concrete operations stage and are quite capable of engaging in arithmetical calculations with a true sense of what a number actually represents. Their ability to engage in more symbolic mathematics involving the manipulation of algebraic quantities representing whole sets of numbers does not come into prominence until age 11 or so, when they have entered the formal operations stage. This ability to manipulate formal mathematical symbols, representing various sets of mathematical objects, continues to grow and expand through adolescence and young adulthood.

However, the higher levels of mathematical ability and the sublime creative productions of mathematicians appear to move beyond Piaget's highest stage of formal operations into what may correspond with Integral Theory's vision-logic level of cognition, where complex inter-relationships are processed symbolically and metaphorically in highly creative ways.[16] This vision-logic level of mathematical cognition is the essential vehicle that allows for the discovery of new mathematical ideas, particularly for highly abstract mathematics that involve combinations of various fields such as Number Theory, Topology, Abstract Algebra, Real and Complex Analysis, and Projective Geometry (see figure 1). At the same time, this vision-logic mathematics allows for the highly theoretical logical proofs of some of the key theorems of applied mathematical disciplines, such as Calculus and Analysis and Statistics (see figure 1). The actual application of mathematics to real-world events largely involves formal-operational to vision-logic cognition, and is exemplified in applications like Statistics, Calculus, Differential Equations, and most especially in combined mathematics/science fields such as Mathematical Physics and Mathematical Biology.

An even more focused perspective on the mathematical line of Integral Theory can be seen from the work of Howard Gardner on Multiple Intelligences.[17] For Gardner, the logical-mathematical line is one type of intelligence, in addition to the linguistic, musical, spatial, bodily-kinesthetic, and personal (inner and outer-directed awareness) intelligences. Although there are various relationships amongst these diverse intelligences, the main features of Gardner's logical-mathematical intelligence include: the crucial importance of the discovery of number; and the hierarchical development from the concept of "number" to the creation of "algebra" (where numbers are regarded as a system and variables are introduced to represent numbers) to the more general concept of "functions" (where one variable has a systematic relation to another variable). Functions may involve real values such as length, width, and time, but may also involve non-real quantities such as imaginary numbers, functions of functions, and significantly more complicated abstractions as well.[18] The two examples I have given from Trigonometry and Calculus (see figures 2 and 3) are examples of functions of real values. As we will see, Perfect Numbers are an example of the concept of number represented in algebra. And the group theory/consciousness example in the Appendix is an example of a more abstract formulation of functions, though applied to the real world in the context of shifts into higher levels of consciousness through the practice of meditation.

The above discussion of Piaget and Gardner for the cognitive-mathematical line can be put into the context of Integral Mathematics perspective #2. From this perspective, essentially any kind of mathematical thinking, whether it is computational or symbolic, would be disclosed by way of an UL quadrivium. The production of written mathematical language and symbols to describe this mathematical thinking would be placed in the UR quadrivium. Teaching and learning (through social interaction) of mathematical ideas and skills would be the crux of the LL quadrivium, encompassing all our interpersonal and interactive educational settings. Using textbooks, calculators, computers, external classroom settings, and verbal mathematical communications would be aspects of the LR quadrivium. Thus, we see how Integral Mathematics situates the mathematical line of development in the context of a four-quadrant analysis.

The Four-Quadrant Mathematics of Perfect Numbers

We come now to our primary example of Integral Mathematics perspective #3: mathematics as a discipline and particular subject of study, which will be taken from the area of Recreational Number Theory and will involve the topic of Perfect Numbers. This topic is also an excellent illustration of how the world of pure mathematics can be introduced to upper elementary school children, illustrating Integral Mathematics perspective #4 (a four-quadrant analysis applied to a particular age group).

The topic of Perfect numbers is a magnificent example of an enticing, unsolved problem in mathematics that can be easily understood by children, the formulation of which involves very large prime numbers that can shed light on an application to government security codes. Let us first define a perfect number as any number where all the numbers that divide into it evenly—not including the number itself—add up to the original number. In other words, a perfect number is the sum of its proper divisors. For example, all the numbers that divide into 8 evenly are 1, 2, and 4. The proper divisors of 8 add up to 7 and therefore 8 is not a perfect number. However, 6 has proper divisors 1, 2, and 3, and they add up to 6; therefore 6 is the first perfect number. With a little bit of diligence, we can determine that the second perfect number is 28, as the proper divisors of 28 are 1, 2, 4, 7, and 14, which indeed add up to 28. But there is an interesting pattern for perfect numbers that we can explore in the context of Recreational Number Theory, and the discovery of this pattern is a good example of the inner creativity of the UL quadrant.

Notice how the first two perfect numbers, 6 and 28, can be written as 6 = 2 X 3 and 28 = 4 X 7. A possible pattern for the third perfect number might therefore be 2 X 3, 4 X 7, and 8 X 11 = 88, where 8 is obtained by doubling 4, and 11 is obtained by adding 4 to 7. Another possible pattern could be 2 X 3, 4 X 7, 8 X 21 = 168, where 21 = 3 X 7, or 2 X 3, 4 X 7, 16 X 11 = 176, where 16 is 4 X 4 (after observing that 4 = 2 X 2), etc. Eventually, with some helpful hints, we will understand the pattern 2 X 3, 4 X 7, 16 X 31 = 496, where the first factor is obtained by squares: 2 X 2 = 4, 4 X 4 = 16, and the second factor is obtained by doubling the first factor and subtracting 1 (i.e., 3 = 2 X 2 – 1, 7 = 2 X 4 – 1, 31 = 2 X 16 – 1). It can be readily checked that 496 is truly the third perfect number, as the proper divisors of 496 are 1, 2, 4, 8, 16, 31, 62, 124, and 248. These proper divisors do add up to 496, and there are no perfect numbers between 28 and 496. (This is wonderful multiplication and division skills practice in the UR quadrant for upper elementary school children: see figure 4.)

- First Perfect Number: 6 = 2 X 3; proper divisors are 1, 2, 3, which add up to 6.
- Second Perfect Number: 28 = 4 X 7; proper divisors are 1, 2, 4, 7, 14, which add up to 28.
- Third Perfect Number: 496 = 16 X 31; proper divisors are 1, 2, 4, 8, 16, 31, 62, 124, 248, which add up to 496.

Figure 4. The First Three Perfect Numbers and Their Proper Divisors

However, if we continue this pattern to try to obtain the fourth perfect number, one obtains 256 X 511 = 130,816 since 16 X 16 = 256 and 511 = 2 X 256 – 1. With the use of an ordinary arithmetic calculator (LR activity) or some knowledge of factor trees and prime factorization, it is quite reasonable for liberal arts college students and children in grades 5 and higher to determine that 130,816 is not a perfect number.[19]

What is the correct pattern to find the fourth perfect number? Try doubling the first factor, doubling once again, and subtracting 1 to get the second factor. For example, since the second perfect number is 28 = 4 X 7, we would have 8 X 15 as a candidate for the third perfect number, but it can easily be checked that 8 X 15 = 120 is not a perfect number. But by doing it once more, we obtain 16 X 31 = 496, which is indeed the third perfect number. The crucial observation is that the second factors of the first three perfect numbers are 3, 7, and 31, all of which are prime numbers (recall that a prime number is a number that has no proper divisors other than 1), and the second factor of the false candidate for the third perfect number is 15, which is not a prime number. Continuing this pattern once more results in 32 X 63, which would be rejected since 63 is not a prime number. But the next candidate is 64 X 127 = 8,128, and it is easy to see that 127 is a prime number. It is quite feasible to determine that 8,128 is a perfect number, and it happens to be the fourth perfect number (see figure 5).

- First Perfect Number: 6 = 2 X 3; 3 = 2 X 2 – 1 and 3 is prime.
- Second Perfect Number: 28 = 4 X 7; 7 = 2 X 4 – 1 and 7 is prime.
- Third Perfect Number: 496 = 16 X 31; 31 = 2 X 16 – 1 and 31 is prime.
- Fourth Perfect Number: 8128 = 64 X 127; 127 = 2 X 64 – 1 and 127 is prime.
- Fifth Perfect Number: 33,550,336 = 4,096 X 8,191; 8,191 = 2 X 4096 – 1 and 8,191 is prime.

Figure 5. The First Five Perfect Numbers and Their Correct Patterns

Continuing this process requires a factor tree and prime factorization to determine proper divisors,[20] serious calculator use, and small group collaborative efforts (productive and fun-loving LL activity). Students will eventually discover that the fifth perfect number is 33,550,336 (see figure 5). Does this pattern always work to find perfect numbers? How many perfect numbers are there? As it turns out, the topic of perfect numbers has by no means been completely solved by mathematicians, as we do not know how many perfect numbers there are.

Only 41 perfect numbers have been found using supercomputers at this time. The exponent itself of the prime number involved in the largest known perfect number was found thru intensive LR activity, and it would take 1,400 to 1,500 pages to write out! Although we know that all even perfect numbers do follow the particular pattern we have described and can be readily made into an algebraic formula for college students and middle school children who are learning algebra, we do not know whether or not there exists an odd perfect number (wonderful UL stimulation). The use of extremely large prime numbers in the formula for even perfect numbers is directly related to how government security codes are devised (LR), where tremendously large composite (non-prime) numbers would need to be factored into two very large prime numbers in order to unlock the code.

From this brief illustration of how I teach the topic of Perfect Numbers as a mathematics enrichment activity, we can see how all four quadrants can be applied to mathematics as a discipline and specific subject (Integral Mathematics perspective #3). The intrinsic exploration of number patterns through hypotheses is an UL activity focused on the interior-individual process of creativity. The actual testing of these patterns to see if they result in successful outcomes involves much practice in the concrete skills of multiplication and division (and algebra for older students) and can be viewed as an UR behavioral activity. I generally have students working in small groups collaboratively to both explore ideas for their patterns as well as test them out, which is a LL activity of sharing creative ideas and procedures. Finally, as the possible candidates for number patterns become extremely large, students utilize technology in the form of calculators (or computers for older students), to obtain results regarding the testing of their ideas for patterns, which is a LR activity that utilizes technology currently available in our social institutions and classroom settings.

Concluding Statement

From this brief glimpse into the possible ramifications of including Integral Mathematics as one of the disciplines seeking to take an Integral perspective, we see that there are rich and enticing possibilities to consider and a number of different approaches that can be taken. A four-quadrant analysis can be applied to the whole range of mathematics as a mathematical line of development, based on the research and theories of Piaget and Gardner (Integral Mathematics perspective #2). A four-quadrant analysis can also be applied to mathematics as a particular discipline of study, as seen through well known applied mathematics examples from Trigonometry and Calculus, a pure mathematics example from Recreational Number Theory, and a combined pure/applied abstract mathematics example joining mathematical group theory and levels of consciousness (see Appendix). These two major perspectives of Integral Mathematics have immediate relationships to viewing mathematics toward a specific age group (perspective #4), engaging in mathematical research (perspective #5), and the establishment of a discipline of Integral Mathematics in its own right (perspective #6). All these perspectives for Integral Mathematics originally sprang from the pioneering efforts of Ken Wilber's calculus of indigenous perspectives: a symbolic mathematical language used to describe the multi-dimensional perspectives inherent in human interaction (Integral Mathematics perspective #1).

We thus see that Integral Mathematics has a great deal of potential to take its place among the other areas of study that are being situated within an Integral framework. It is my hope that this article may serve as a call to other mathematicians to furnish their own examples of Integral Mathematics and that such work will be a rich and valuable inclusion in the exciting development of Integral Institute. I would like to form a network of mathematicians who are excited about this endeavor and I welcome hearing about your own ideas and interests concerning what Integral Mathematics means to you and how to get it off the ground.

Appendix
MATHEMATICAL GROUP THEORY AND CONSCIOUSNESS

I will now turn my attention to Abstract Algebra and Group Theory to illustrate how we might use pure mathematics theory to study shifts into higher levels of consciousness. We will apply a mathematical group model to describe the shift from the rational to the vision-logic level of consciousness in the cognitive stream (although one could apply this theory to shifts between different levels in different streams). Specifically, we will show how a group theoretical mathematical model can explain how a particular state of consciousness may have a significant impact on a person developing into a higher level of awareness. For example, if I am in the middle of a continuum between the rational and vision-logic levels of consciousness, prolonged periods of meditation over a certain time period may be a significant factor in enabling me to move closer toward vision-logic. We shall refer to our meditation experience as M mod x, which means that person x is experiencing meditation (M) within him/herself via him/herself. It is understood that we are focusing upon some particular type of meditation represented by M and a particular individual represented by x, as our theory is attempting to capture the subjective individualized framework of both the person and the type of meditation experienced.

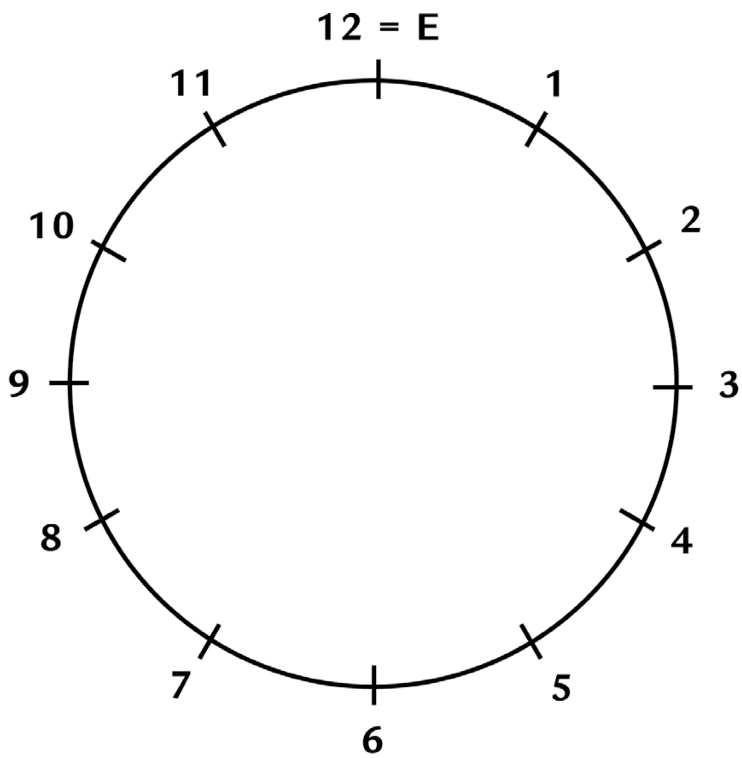
Keep in mind that our theory is at the beginning stages, and we will therefore make rather simplistic assumptions, such as an increase in the number of hours of meditation results in a corresponding increase in moving from the rational to vision-logic level, in order to illustrate the basic mathematical ideas involved. Clearly, this mathematical simplification does not realistically describe the actual phenomenon of how people develop into higher levels of consciousness, as many other factors come into play, not the least of which is that simply increasing the number of hours meditating may not have a corresponding effect of shifting into a higher level of consciousness. The intention of the person meditating may be a significant variable that also should be considered.

Mathematical Group Theory

A mathematical group is defined as a set of elements (S) with an operation (*) such that the following properties are satisfied.

- 1. If x is an element of S and y is an element of S then x*y is also an element of S.
- 2. There is an identity element E in S such that for all elements x in S, x*E = E*x = x.
- 3. For all elements x, y, and z in S, we have x*(y*z) = (x*y)*z (associative law).
- 4. For each element x in S there exists an element y in S such that x*y = y*x = E; we refer to this element y as x^(-1) and call it the inverse element of x.

If for all elements x and y in S we also have the property that x*y = y*x, then our group is referred to as a commutative (or abelian) group. A simple example of a commutative group is the infinite set of integers S = {…-3, -2, -1, 0, 1, 2, 3…} under addition. We see that the sum of two integers is always an integer, zero is the identity element, the associative law holds, for any integer x we have x^(-1) = -x, and for any integers x and y, we have x + y = y + x. Thus the set of integers is a commutative group under addition. If there is an element x in our group S such that every element y in S can be written as x^n for some integer n, where x^n refers to (x*x*x…*x) n times if n > 0, x^(-n) = (x^(-1))^n, and we define x^0 = E, then our group S is referred to as a cyclic group with the generator x. It is an easy mathematical exercise to prove that all cyclic groups are commutative. We see that our infinite commutative group of integers is actually a cyclic group generated by 1. For an example of a finite commutative cyclic group under addition consisting of 12 elements, think of the hour hand clock numbers of an ordinary 12 hour clock (see figure 6). We see that 1 is a generator of the group, 12 is the identity element (12 + 5 = 5, 12 + 7 = 7, 12 + (-4) = 12 + 8 = 8, etc.), and for any hour hand clock number x we have x^{12} = (x + x + x + …x) (12 times) = 12. Given any clock number x, we see that 12 – x is the inverse of x since x + (12 – x) = 12 = the identity element E (5 + (12 – 5) = 5 + 7 = 12 = E, etc.).



$$12 + 3 = 3,\ \ 3 + (12 - 3) = 3 + 9 = 12 = E,$$
$$12 + x = x,\ \ x + (12 - x) = 12 = E,$$
$$3^{12} = (3 + 3 + 3 + \ldots 3)\ (12\ \text{times}) = 12 = E,\ \ x^{12} = E$$

Figure 6. Clock Arithmetic Example

Biquasi-Groups and Consciousness Shifts

To formulate a group theoretical model of shifts into higher levels of consciousness, we shall define the transition into the next higher as a “biquasi-identity element” I of a “biquasi-group” (these "biquasi" terms will soon be defined). Thus in our above meditation example, the transition from the rational to the vision-logic level of consciousness is what we define as the biquasi-identity element I in what turns out to be a cyclic biquasi-group S generated by M mod x. We shall interpret the equation $(M \text{ mod } x)^6 = I$ to mean that continued practice of our meditation over a certain time period will be a significant factor in enabling person x to move from the rational to the vision-logic level of consciousness. According to Integral Theory, repeated contact with altered states of consciousness, such as meditation, may help a person disidentify from their current level of consciousness, thereby enabling a person to take as object that which was formerly a subject for them. Or, as Wilber puts it, transformation involves disidentification with one's current stage, identification with the next higher stage, and integration of aspects of the previous stages into the higher stage.[21] Mathematically, we can think of our meditation example as resembling a finite cyclic group S consisting of 6 elements generated by M mod x. We say "resembling" as opposed to "actual" due to the biquasi nature of our group, which we shall now describe.

For illustrative purposes, let's assume that M mod x denotes 30 hours of meditation over the time period of one month. According to our group theoretical model (in its preliminary and simplistic form), person x would make significant progress toward moving into the vision-logic level of consciousness if he or she were to diligently continue this meditation practice for 180 hours over a 6 month time period, which is the meaning of the equation $(M \text{ mod } x)^6 = I$. We have the equations $M \text{ mod } x + M \text{ mod } x = (M \text{ mod } x)^2$ (meaning simply that 30 hours + 30 hours equals 60 hours of meditation, which has twice the impact as 30 hours of meditation on person x's potential consciousness shift, once again given our extreme mathematical simplification), $(M \text{ mod } x)^2 + (M \text{ mod } x)^3 = (M \text{ mod } x)^5$, etc.

In general, we can say $(M \text{ mod } x)^m + (M \text{ mod } x)^n = (M \text{ mod } x)^{(m+n)}$ for positive integers m and n. However, once person x reaches the vision-logic level of consciousness, the impact of their meditation practice is no longer in the same way relevant to progressing to the next higher level of consciousness (illumined mind, in this case), as we now acknowledge mathematically that other factors may come into the picture. We therefore define I*I = I and I*(M mod x)^n = (M_0 mod x)^n, where M_0 mod x means that person x is now meditating while at a vision-logic level of consciousness. Note that by using this scheme, we would have I*(M mod x)^2 = I*I*(M mod x)^2 = (M_0 mod x)^2 and I*(M mod x)^62 = I*(M mod x)^2 = (M_0 mod x)^2. We therefore make no further interpretation of the impact of the meditation experience on person x once the vision-logic is reached. Certainly another mathematical scheme could be devised, but at this point we are merely illustrating the essential idea in its simplest formulation.

At any rate, our equation I*(M mod x)^n = (M_0 mod x)^n resembles the requirements for I to be an identity element of a group, but there is a major problem in that M becomes M_0 (signifying that we are now in the vision-logic level of consciousness). This change from M to M_0 does necessitate us calling our identity element something resembling (but not quite exactly) an identity element; we shall call it a “biquasi-identity element.” The term “biquasi” refers to the fact that we are using a second set to refer to our shift into vision-logic. Of course, this same general idea could be applied to shifts into any of the various levels of consciousness. In a similar manner, we don’t quite have a bona fide group, but if we use this biquasi-identity element in the required properties of a group, we find that our group properties are essentially satisfied, and we now have what we shall refer to as a “biquasi-group.” More specifically, we have a “cyclic biquasi-group” generated by M mod x.[22]

Biquasi-Group Model Applied to the Four Quadrants

In my paper, “On the 2-Class Field Tower of Some Imaginary Biquadratic Number Fields,” I describe and compare biquasi-groups that represent various degrees of meditation and yoga practices that can occur across various streams, with their resulting transformative effectiveness.[23] But for our present purposes, it is most relevant that the mathematical group theory used in my paper is at the heart of pure mathematics (the UL). On the other hand, the application of this pure mathematical group theory to Integral Theory represents the world of applied mathematics: an UR activity focusing on the external behavioral equations derived from the intrinsic, UL mathematical model. The application of this mathematical model could involve a variety of communities of meditators and yoga practitioners engaged in a discipline of spiritual activity, thereby entering the LL. Finally, one could significantly increase both the number and relationship of variables utilized to make the actual phenomenon being investigated more realistic, plus increase the combinations of spiritual disciplines and various streams investigated, to the point where computer software programs would be needed to work effectively with the mathematical equations, thereby engaging the LR.

Endnotes

[1] See Wilber, *Sex, ecology, spirituality: The spirit of evolution*, 1995; *A brief history of everything*, 2000a; and *Integral psychology: Consciousness, spirit, psychology, therapy*, 2000b
[2] King, *The art of mathematics*, 1992
[3] Wilber, "Appendix B. An integral mathematics of primordial perspectives," 2004
[4] See Gardner, *Frames of mind: The theory of multiple intelligences*, 1983; Lakoff & Nunez, *Where mathematics comes from*, 2000; and Piaget, *The child's conception of number*, 1952
[5] See Wilber, *Sex, ecology, spirituality: The spirit of evolution*, 1995; *A brief history of everything*, 2000a; and *Integral psychology: Consciousness, spirit, psychology, therapy*, 2000b
[6] Benjamin, "On the 2-class field tower of some imaginary biquadratic number fields," 2006
[7] However, I must fully admit that my interest in mathematics is primarily for its inner artistic beauty. Upon studying Ken Wilber's Calculus of Indigenous Perspectives, I felt the inclination to write an essay in which mathematical group theory—the basic logical structure of the pure mathematics discipline of Abstract Algebra—could be applied to shifts into higher levels of consciousness. See Benjamin, "A group theoretical mathematical model of shifts into higher levels of consciousness in Ken Wilber's integral theory," 2003. I view the ideas in this theory as a step towards a unification of pure mathematics (UL) and applied mathematics (UR), or an integration of the Upper-Left and Upper-Right quadrants for the mathematical stream. The mathematical model for this group theory/consciousness example is included in the Appendix for interested readers, as the mathematics does require a high degree of concentration to follow.
[8] Benjamin, *Numberama: Recreational number theory in the school system*, 1993
[9] See Benjamin, *Number theory, developmental mathematics, and the discovery approach*, 1990; *Numberama: Recreational number theory in the school system*, 1993; and "Number theory—An igniter for mathematical sparks," 2000
[10] See Benjamin, *Numberama: Recreational number theory in the school system*, 1993
[11] See Benjamin, *Numberama: Recreational number theory in the school system*, 1993, for details.
[12] Wilber, "Appendix B. An integral mathematics of primordial perspectives," 2004
[13] Piaget, *The child's conception of number*, 1952
[14] Piaget, *The child's conception of number*, 1952
[15] Dehaene, *The number sense: How the mind creates mathematics*, 1997
[16] Lakoff & Nunez, *Where mathematics comes from*, 2000
[17] Gardner, *Frames of mind: The theory of multiple intelligences*, 1983
[18] Gardner, *Frames of mind: The theory of multiple intelligences*, 1983
[19] See Benjamin, *Numberama: Recreational number theory in the school system*, 1993, for details.
[20] Benjamin, *Numberama: Recreational number theory in the school system*, 1993
[21] Wilber, *Sex, ecology, spirituality: The spirit of evolution*, 1995; *A brief history of everything*, 2000a; and *Integral psychology: Consciousness, spirit, psychology, therapy*, 2000b
[22] See Benjamin, "On the 2-class field tower of some imaginary biquadratic number fields," 2006, for a more formal mathematical definition of a biquasi-group, along with a few basic lemmas (results that have been mathematically proven but are not as significant as theorems) that describe some of its properties.
[23] Benjamin, "On the 2-class field tower of some imaginary biquadratic number fields," 2006

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